ARIZONA STATE UNIVERSITY

Tempe, Arizona

COLLEGE of ENGINEERING SCIENCES

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# SEMI-ANNUAL TECHNICAL PROGRESS REPORT

(Period: August 1, 1966 to January 31, 1967)

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# "INVESTIGATION OF DYNAMIC BEHAVIOR OF THIN SPHERICAL SHELLS"

Principal Investigator: James P. Avery

## A. REVIEW OF FIRST YEAR EFFORT:

This investigation concerns the development of a Green's Function approach to the vibration problem of thin spherical shell segments. The first year effort culminated in the formulation of the analytical approach to be used, the development of the governing equations and the methods of solution. The results were presented in the first year annual report published as NASA Contractor Report 602 (September 1966). The analytical approach, in brief, consists of the development of a set of integral equations governing a static problem equivalent to the desired vibration problem.

The solutions of two fundamental elastostatic problems are used in superposition to obtain the influence functions (or Green's Functions) for the set of governing integral equations. The two fundamental problems are respectively, (I) that of a complete thin spherical shell on an elastic foundation subjected to a unit normal load and (II) that of a complete thin spherical shell on the same elastic foundation subjected to a unit tangential load. The governing equations for each of these fundamental problems were developed and reduced to simple form. Methods of solution were indicated. The report, in addition, outlined the manner in which the fundamental problem solutions were to be combined to obtain the desired Green's Functions for a specific boundary value (vibration) problem. Finally, the report suggested a numerical solution for the resulting integral equations.

## B. SUMMARY OF PROGRESS:

During the first half of the second year effort, numerical solutions were attempted for the fundamental problems I and II employing the method of solution developed earlier and outlined in the annual report. Serious shortcomings were uncovered in attempting to follow this solution formulation. These difficulties stemmed in part from an erroneous assumption of negligible relative magnitude of the meridianal displacement, v, and also from convergence difficulties (in certain ranges of the independent variable) rendering numerical superposition of solutions untenable.

A new problem solution formulation was then evolved for both fundamental problems and was successfully applied to obtain numerical solutions for these fundamental problems. The new formulation is based upon a solution in terms of Bessel's Functions and Kelvin Functions in the neighborhood of the load singularity. The Kutta-Merson process of numerical integration used to obtain complete numerical solutions, which solutions are superimposed to match the appropriate boundary conditions. A detailed discussion of this formulation follows. Plots of resulting solutions for sample geometric parameters and foundation modulus appear in Appendix I.

## C. FUNDAMENTAL PROBLEM I, SOLUTION FORMULATION:

For the symmetric bending of the thin spherical shell on an elastic foundation, the differential equations of equilibrium and the elastic law may be combined and reduced to the following two equations (corresponding to equations (11) and (12b) of the Annual Report):

$$L\ (Q) + \nu Q = (1 - \nu^2 + \beta)\ \bar{\chi} - (2 + \nu)\beta\ \bar{v} \tag{1}$$

$$L\ (\bar{\chi}) - \nu\bar{\chi} = -\frac{Q}{\alpha} \tag{2}$$

where symbols are defined as in earlier reports except (with the spherical radius taken as unity) new symbols are:

$$\bar{v} = Kv$$

$$\bar{w} = Kw$$

$$\bar{\chi} = \bar{v} + \mathring{\bar{w}}$$

Additionally, eliminating $\bar{w}$ and the membrane stress resultants between the equations of the elastic law and the first two equilibrium equations, one obtains,

$$L\ (\bar{v}) + (1 - \beta)\bar{v} = Q + (1 + \nu)\ \bar{\chi} \tag{3}$$

Finally, solving the first two equations of the elastic law for $\bar{w}$ and then eliminating the membrane stress resultants (using equilibrium relations) leads to,

$$\bar{w} = \frac{1}{\beta + 2\ (1 + \nu)}\ \frac{\{[(1 + \nu)\ \bar{v} + Q]\ \sin\phi\}^{\circ}}{\sin\phi} \tag{4}$$

which permits a direct evaluation of $\bar{w}$, when $\bar{v}$, $Q$, $\mathring{\bar{v}}$ and $Q^{\circ}$ are known.

The strategy of solution of the three simultaneous equations (1) (2) and (3) in $Q$, $\bar{\chi}$ and $\bar{v}$ consists of the following five steps:

(1) Obtain an approximate starting solution, consistent with the load singularity, for small $\phi$ (which approximation approaches exactness as $\phi$ approaches zero).

(2) Employ the Kutta-Merson numerical integration process using the starting solution as initial values to obtain a complete solution for the spherical cap (that is, for $\phi$ up to $\pi/4$ radians).

(3) Obtain three additional independent solutions (using the Kutta-Merson method) for the spherical cap, each associated with different initial conditions but with no load singularity.

(4) Obtain six independent solutions for the girth zone (again using the Kutta-Merson process), where the variable $\phi$ extends from $\pi/2$ to $3\pi/4$ radians.

(5) Combine the above ten solutions employing symmetry principles to obtain the solution to the complete spherical shell on the elastic foundation.

STEP 1

Near the load singularity it is tentatively assumed (and may subsequently be verified) that the quantity $\bar{v}$ is of negligible order in comparison to $\bar{\chi}$. Hence, ignoring the last term in equation (1), upon substitution of equation (2) into (1) we obtain,

$$L^2 \bar{\chi} + 4 \varkappa^4 \bar{\chi} = 0$$

where

$$4 \varkappa^4 = \frac{1 - \nu^2 + \beta - \nu^2 \alpha}{\alpha}$$

If then $\phi$ is very small the operator L becomes approximately:

$$L = (\ )^{\circ\circ} + \frac{1}{\phi}(\ )^{\circ} - \frac{1}{\phi^2}(\ )$$

The resulting fourth order equation then has solutions in terms of Kelvin Functions:

$$\bar{\chi} = A_1 \, \mathrm{ker}'x + A_2 \, \mathrm{kei}'x$$

where

$$x = \sqrt{2}\, \varkappa \, \phi$$

From symmetry $\bar{\chi}$ must vanish at the origin, hence

$$\bar{\chi} = A_2 \text{ kei}'x$$

Using the additional boundary condition that the shear Q must satisfy the load singularity requirement,

$$\lim_{\phi \to 0} (2\pi\phi Q) = -1$$

and the condition that $\bar{w}$ (evaluated by equation 4) must be finite, the following solution results:

$$Q = \frac{\mathcal{H}}{\sqrt{2}\pi} \left[\text{ker}'x - \frac{1+\nu}{2\mathcal{H}^2} \text{ kei}'x\right]$$

$$\bar{\chi} = \frac{-1}{2\sqrt{2}\pi \; \alpha \mathcal{H}} \text{ kei}'x$$

$$\bar{v} = \frac{1}{2\sqrt{2}\pi \mathcal{H}} \left[\text{kei}'x + \frac{1+\nu}{2\mathcal{H}^2\alpha} (\text{ker}'x + \frac{1}{x})\right]$$

$$\bar{w} = \frac{-1}{4\pi\mathcal{H}^2\alpha} \text{ kei } x$$

STEP 2

Introducing the symbols,

$$Y_1 = \frac{(\bar{v} \sin \phi)^\circ}{\sin \phi} \qquad Y_2 = \bar{v}$$

$$Y_3 = \frac{(Q \sin \phi)^\circ}{\sin \phi} \qquad Y_4 = Q$$

$$Y_5 = \frac{(\bar{\chi} \sin \phi)^\circ}{\sin \phi} \qquad Y_6 = \bar{\chi}$$

the equations (1), (2) and (3) may be rewritten as

$$\begin{aligned}
Y_1^\circ &= (\beta - 2)\, Y_2 + Y_4 + (1 + \nu)\, Y_6 \\
Y_2^\circ &= Y_1 - Y_2 \cot \phi \\
Y_3^\circ &= -\,(2 + \nu)\beta\, Y_2 - (1 + \nu)\, Y_4 + (1 - \nu^2 + \beta)\, Y_6 \\
Y_4^\circ &= Y_3 - Y_4 \cot \phi \\
Y_5^\circ &= -\frac{1}{\alpha} Y_4 - (1 - \nu)\, Y_6 \\
Y_6^\circ &= Y_5 - Y_6 \cot \phi
\end{aligned} \qquad (5)$$

Given the initial conditions obtained from Step 1, the set of equations may be solved numerically in an efficient manner by the Kutta-Merson method.

STEP 3

Three other solutions of equations (5) may be obtained with three independent sets of initial conditions. First corresponding roughly to a membrane solution we have the initial values (up to third order terms in $\phi$) which consistently satisfy the equations (5).

$$Y_2 = \frac{\phi}{2} + \frac{\phi^3}{4}$$

$$Y_4 = -(2 + \nu)\beta \frac{\phi^3}{8}$$

$$Y_6 = \frac{\phi^3}{8}$$

Secondly, letting the shear resultant, Q, dominate we have the consistent set of initial conditions,

$$Y_2 = 0$$

$$Y_4 = \frac{\phi}{2}$$

$$Y_6 = 0$$

Thirdly, letting the bending moment terms dominate we obtain a consistent set of initial conditions,

$$Y_2 = \frac{\phi^3}{8}$$

$$Y_4 = (1 - \nu^2 + \beta) \frac{\phi^2}{4}$$

$$Y_6 = \frac{\phi}{2}$$

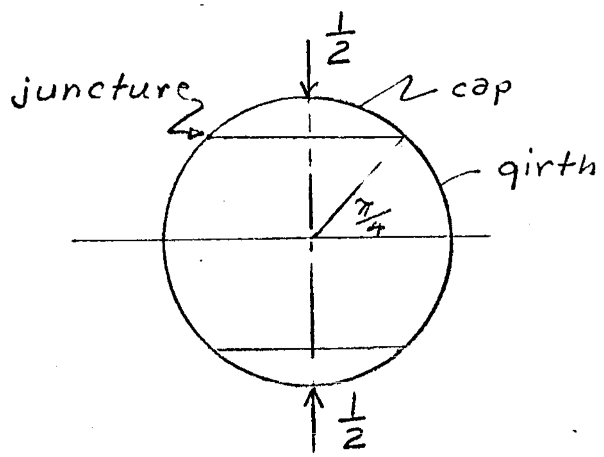
Using these three sets of initial conditions, three independent solutions of equations (5) are obtained by the Kutta-Merson numerical integration for the range of $\phi$ up to $\pi/4$ radians.

STEP 4

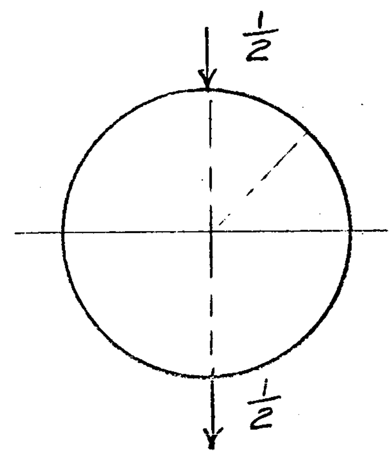
Six independent solutions for the girth zone are obtained by letting each of the functions $Y_1$ through $Y_6$ be non-zero independently for the initial conditions. It should be noted that this gives rise to three solutions that are symmetric with respect to the equator and three that are anti-symmetric, as the initial conditions are expressed at the equator.

STEP 5

Consider the resultant fundamental problem I to be resolved into symmetric and anti-symmetric component problems as illustrated below:




Symmetric




Anti-symmetric

The symmetric problem consists of the four cap solutions of steps (2) and (3) in superposition matching boundary conditions (at the juncture) with the superposed three symmetric girth solutions of Step (4). Similarly, the anti-symmetric problem consists of the four cap solutions in superposition made to match boundary conditions (at the juncture) of the superimposed three anti-symmetric girth solutions. Finally the resulting symmetric and anti-symmetric component problem solutions are added to obtain the complete solution for fundamental problem I. Computed results for sample geometric and foundation conditions appear in Appendix I.

D. FUNDAMENTAL PROBLEM II, SOLUTION FORMULATION

For the non-axisymmetric problem of the thin spherical shell on the elastic foundation, we introduce the following dependent functions:

$$\tau = (\bar{u} \sin \phi)^\circ + \bar{v}$$

$$\bar{u}, \ \bar{v}, \ \bar{w}$$

where

$$(\ )^\circ = \frac{d(\ )}{d\phi}$$

$$Ku = \bar{u} \sin \theta$$

$$Kv = \bar{v} \cos \theta$$

$$Kw = \bar{w} \cos \theta$$

with $\phi$ and $\theta$ as the spherical polar angles. Consistent with the symmetry for the unit tangent load problem, the angle $\theta$ is measured from the meredianal plane containing the unit load.

Governing differential equations in the functions $\tau$ and $\bar{u}$ are obtained from a combination of the equilibrium equations and the elastic law for the non-axisymmetric problem. Upon simplification and reduction these equations (corresponding to equations (27) and (29) of the Annual Report) become:

$$\tau^{\circ\circ} - \tau^\circ \cot \phi - \rho\tau = 0 \tag{6}$$

$$J\left(\bar{u} \sin \phi - \frac{\alpha}{1+\alpha} \bar{w}\right) = \left[\frac{\beta}{1+\alpha} - (1-\nu)\right]\bar{u} \sin \phi + \frac{1+\nu}{2} \tau^\circ + \frac{3-\nu}{2} \tau \cot \phi - \left[(1+\nu) - \frac{2\alpha}{1+\alpha}\right]\bar{w} \tag{7}$$

where the operator J( ) is defined as,

$$J(\ ) = \left\{\frac{[(\ ) \sin \phi]^\circ}{\sin \phi}\right\}^\circ$$

and

$$\rho = \frac{2\beta}{(1-\nu)(1+\alpha)} - 2.$$

Introducing new symbols for dependent functions,

$$Y_2 = \frac{\tau}{\sin \phi}$$

$$Y_1 = Y_2^\circ + Y_2 \cot \phi$$

$$Y_4 = \bar{u} \sin \phi - \frac{\alpha}{1+\alpha} \bar{w}$$

$$Y_3 = Y_4^\circ + Y_4 \cot \phi$$

equations (6) and (7) are expressible as four first order equations:

$$Y_1^\circ = \rho\, Y_2 \tag{8a}$$

$$Y_2^\circ = Y_1 - Y_2 \cot\phi \tag{8b}$$

$$Y_3^\circ = \frac{1-\nu}{2}\rho\, Y_4 + \frac{1+\nu}{2} Y_1 \sin\phi + \frac{3-\nu}{2} Y_2 \cos\phi \tag{8c}$$
$$-[(1+\nu) - \frac{\alpha}{1+\alpha}(\frac{1-\nu}{2}\rho + 2)]\bar{w}$$

$$Y_4^\circ = Y_3 - Y_4 \cot\phi \tag{8d}$$

It should be noted that the function $\bar{w}$ is known from the solution of fundamental problem I since by Betti's reciprocal theorem the displacement function $\bar{v}$ of problem I is equal to $-\bar{w}$ of problem II.

A solution of equations (8) for the complete sphere, consistent with the load singularity is obtained in following several steps similar to those for problem I.

(1) Obtain an approximate starting solution consistent with the load singularity valid for small $\phi$.

(2) Ingrate numerically (using the Kutta-Merson method) the equations (8) employing the starting solution as initial conditions, to obtain a solution for the hemisphere.

(3) Obtain two additional solutions for the hemisphere with two independent sets of initial conditions for the no load singularity condition.

(4) Combine the above three solutions employing symmetry principles to arrive at a solution in superposition for the complete sphere on the elastic foundation.

STEP 1

Near the load singularity the variable $\phi$ is small and only the first term of the Taylor's series for sin $\phi$ and for cos $\phi$ are retained. Consequently equation (6) becomes:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - (1 + \frac{1}{x^2})\, y = 0 \tag{9}$$

where

$$y = \left(\frac{\tau}{\sin\phi}\right) \text{ and } x = \sqrt{\rho}\,\phi$$

Equation (9) has the solutions

$$y = I_1(x),\ K_1(x) \qquad \text{(Bessel's Functions)}$$

The solution $K_1(x)$ is retained:

$$\tau = -AK_1(x)$$

Ignoring the function $\bar{w}$ near the pole (the validity of which is subsequently verified) and introducing the solution for $\tau$, equation (7) becomes:

$$\frac{d^2Z}{ds^2} + \frac{1}{s}\frac{dZ}{ds} - Z\left(1 + \frac{1}{s^2}\right) = \left[\frac{1+\nu}{1-\nu} x\, K_o(x) - \frac{3-\nu}{1-\nu} K_1(x)\right] \frac{A}{\sqrt{\rho}} \qquad (10)$$

where

$$Z = \bar{u}\sin\phi \text{ and } s = \sqrt{\frac{1-\nu}{2}\rho}\;\phi$$

Combining a particular solution of equation (10) with the complimentary solution one may obtain:

$$Z = \frac{A}{\rho}\,\frac{3-\nu}{1-\nu}\sqrt{\frac{1-\nu}{2}\rho}\left[\frac{1}{s} - K_1(s)\right]$$

which contains only a logarithmic singularity. From this, for small $\phi$, we obtain the dominant term for $\bar{u}$,

$$\bar{u} = -\frac{A(3-\nu)}{4}\log(s)$$

and hence for $\bar{v}$ (considering the definition of $\tau$)

$$\bar{v} = A\left[\frac{(3-\nu)}{4}\log(s) - \frac{1+\nu}{4}\right]$$

Near the load, the equilibrium boundary condition reduces to

$$1 = \int_0^{2\pi} (N_{\phi\theta}\sin\theta - N_\phi\cos\theta)\sin\phi\, d\theta\,.$$

Substituting the elastic law for the membrane stresses and the solutions for $\bar{u}$ and $\bar{v}$, the constant A is evaluated,

$$A = -\frac{1}{\pi(1-\nu)}$$

thus providing a solution valid near the load singularity.

STEP 2

The initial values for $Y_1$, $Y_2$, $Y_3$, and $Y_4$ for a small value of $\phi$ may be obtained directly from the starting solution. The Kutta-Merson procedure is then employed to carry the solution to $\phi = \pi/2$ radians.

STEP 3

Two additional independent solutions for the hemisphere are obtained from two independent sets of initial conditions.

In one case assuming a finite initial value for $N_\phi$, a consistent solution for small $\phi$ is (considering low order terms):

$$Y_1 = 2$$

$$Y_2 = \phi$$

$$Y_3 = -2\,\frac{5-\nu}{1-\nu}\,\frac{1}{\rho}$$

$$Y_4 = Y_3\,\frac{\phi}{2}$$

A second solution is obtained for zero initial $N_\phi$:

$$Y_1 = 0$$

$$Y_2 = 0$$

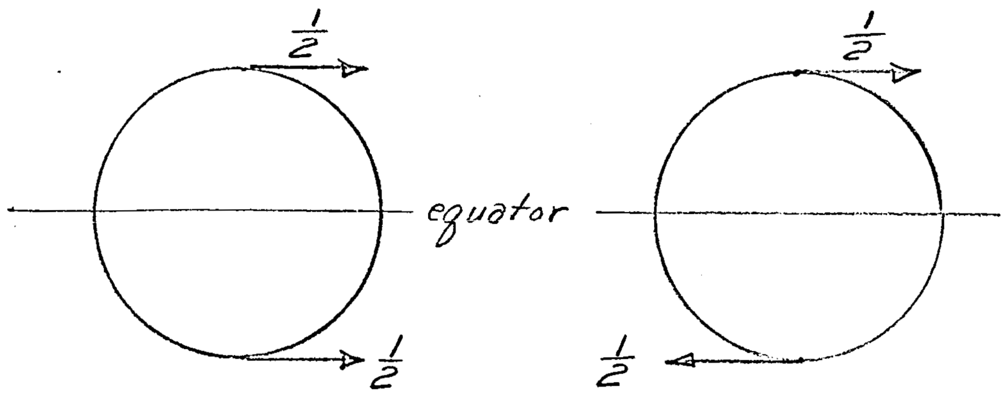
$$Y_3 = 2$$

$$Y_4 = \phi$$

Two independent solutions result from numerical integration using the above two sets of solutions as initial conditions.

STEP 4

The complete sphere problem under the action of a unit tangential load may be resolved into two component problems, one symmetric with respect to the equatorial plane,the other anti-symmetric.




The boundary conditions at the equator for the symmetric problem are:
$\bar{v} = 0$ and $\bar{u}° = 0$.

For the anti-symmetric case the equatorial boundary conditions are
$\bar{v}° = 0$ and $\bar{u} = 0$.

Each of the component problems may now be solved independently by superposition of solutions from Steps (2) and (3) to match the appropriate equatorial boundary conditions. Then finally, the two component problems are superimposed to obtain the solution to fundamental problem II. Computed results for sample geometric and foundation conditions appear in Appendix I.

# APPENDIX I

For the numerical integration of the equations (5) and the equations (8) and for subsequent computations, numerical values are required for the dimensionless parameter $\alpha$, $\beta$, and $\nu$. These quantities were defined earlier as follows:

$$\alpha = \frac{h^2}{12a^2}$$

$$\beta = \frac{ka^2}{K}$$

$\nu$ = Poisson's ratio

where

$h$ = shell thickness

$a$ = radius of shell mid-surface

$k$ = elastic foundation modulus, force per unit area per unit deflection

$K$ = membrane stiffness constant

$$K = \frac{Eh}{1 - \nu^2}$$

$E$ = Young's modulus

For a sample computation the parameters $\alpha$, $\beta$ and $\nu$ were taken as:

$\alpha = .0001$

$\beta = 15.09$

$\nu = .3$

The governing differential equations were integrated, as outlined earlier in the report, and the functions $\bar{w}$, $\bar{v}$, and $Q_\phi$ were computed for the fundamental problem I. The results are plotted as functions of the polar angle $\phi$ in figures (1), (2) and (3). The differential equations for the fundamental problem II were also integrated permitting computation of the functions $\bar{u}$ and $\bar{v}$ for problem II. These results appear as curves shown in figures (4) and (5).

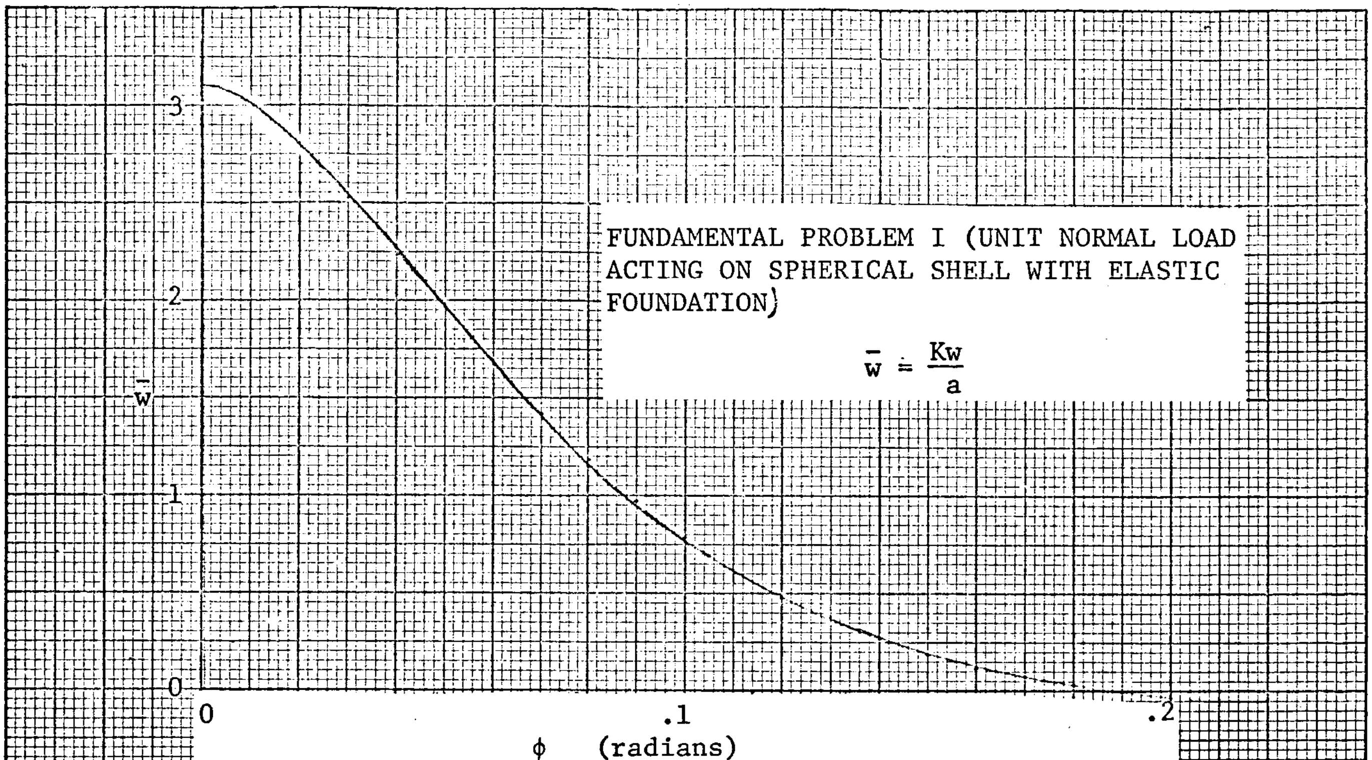



Figure 1. Radial Displacement as a Function of $\phi$.

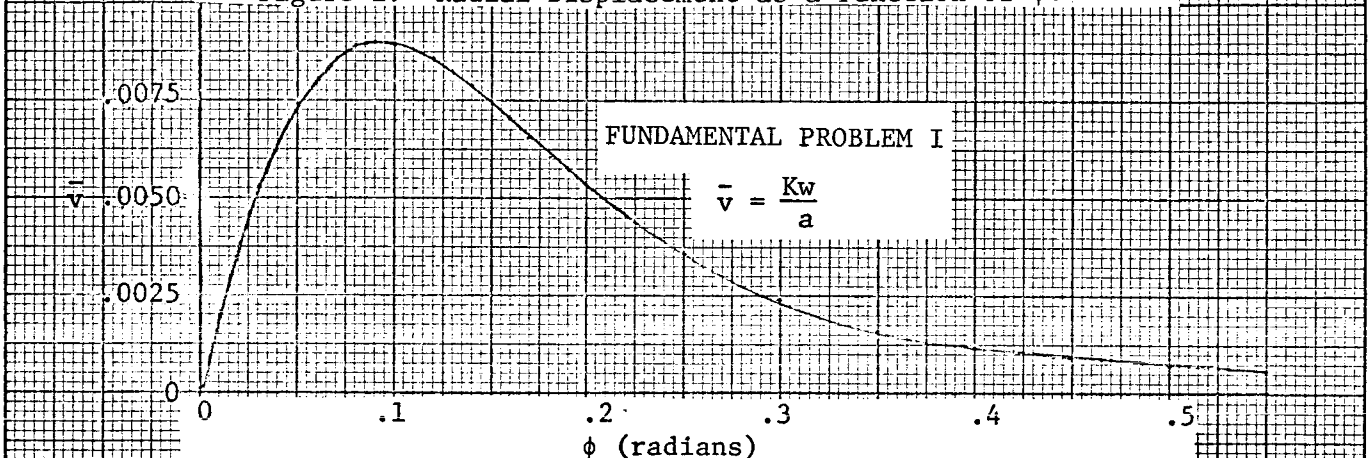



Figure 2. Tangential Displacement as a Function of $\phi$.

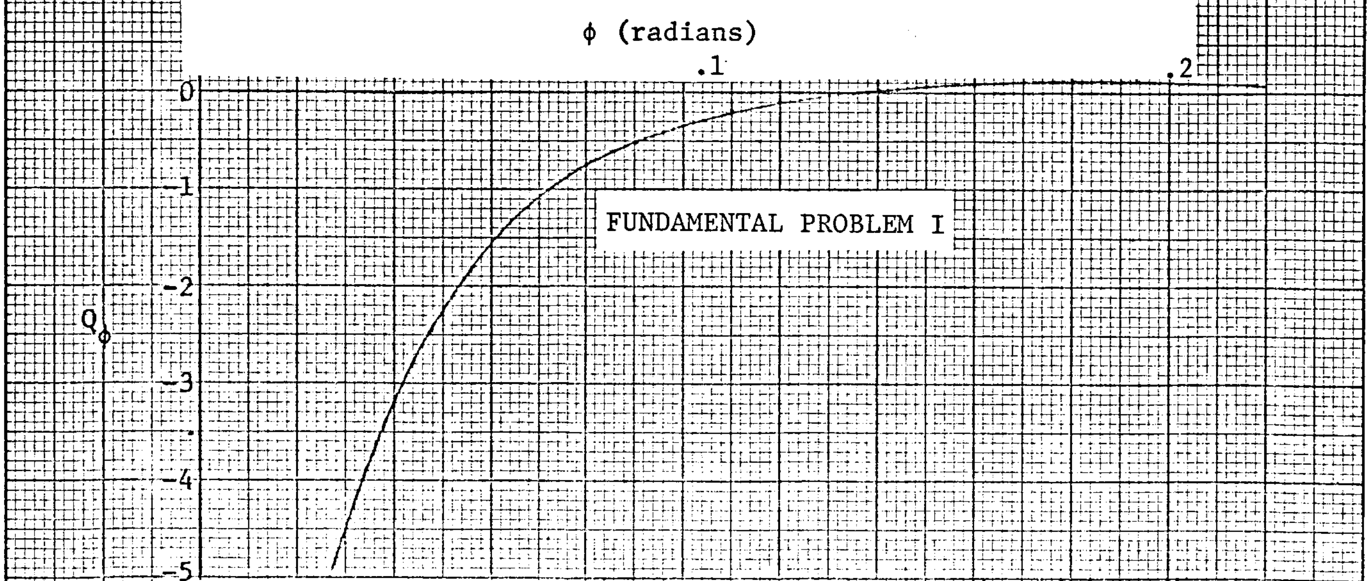



Figure 3. Shear Stress Resultant as a Function of $\phi$.

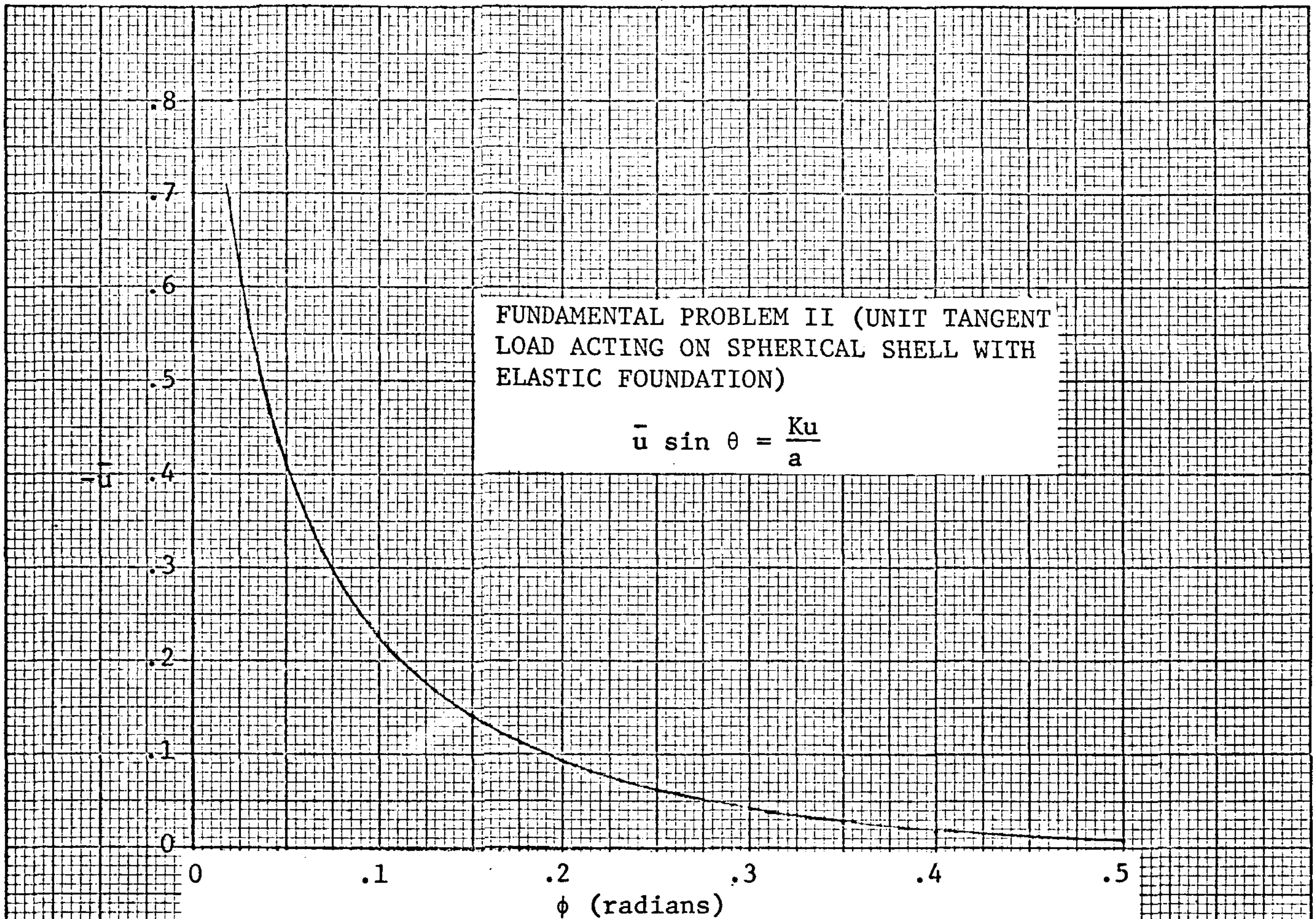



Figure 4. Latitudinal Displacement as a Function of ϕ.

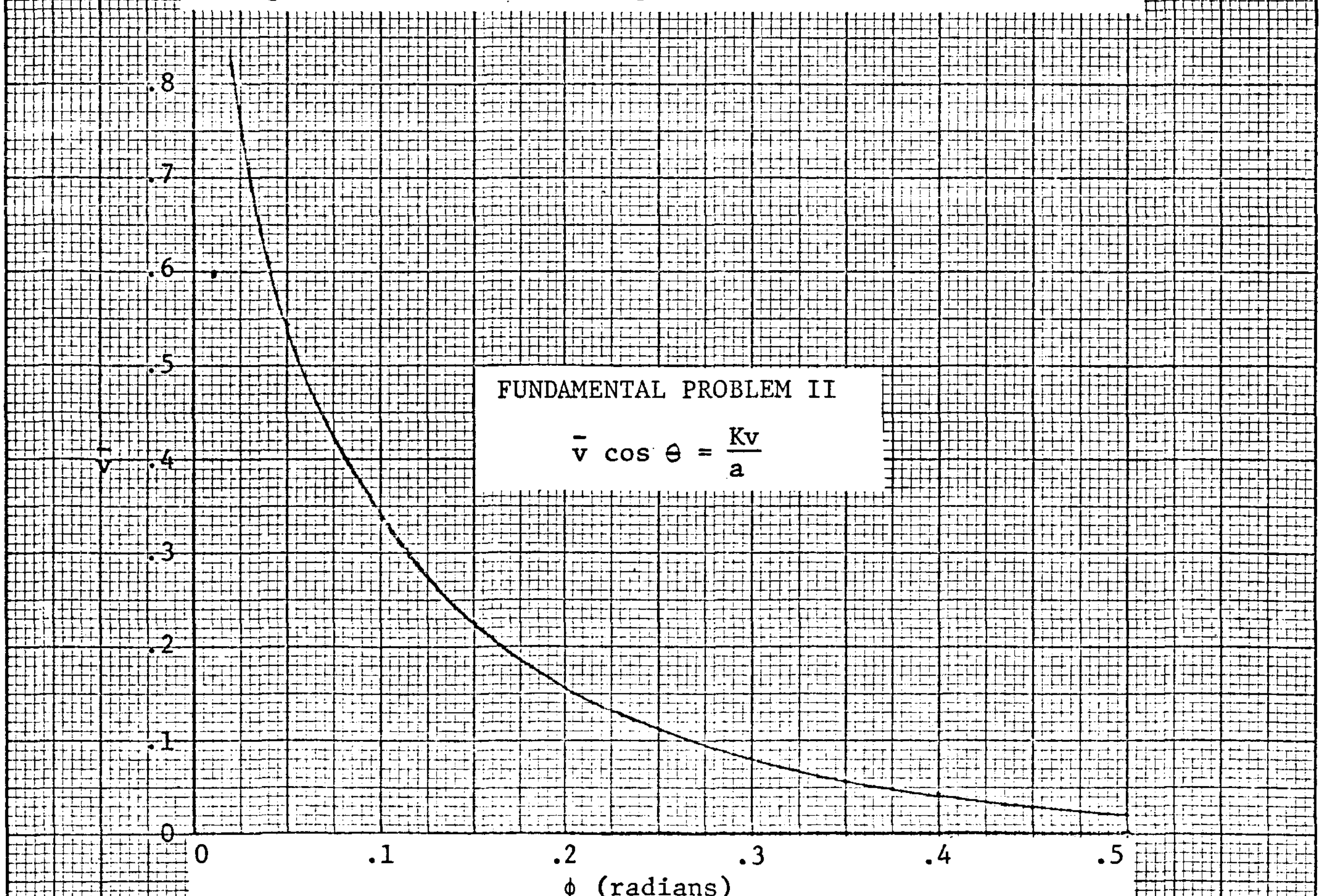



Figure 5. Meridianal Displacement as a Function of ϕ.